ME 141
Engineering Mechanics

Lecture 3: Rigid bodies: Equivalent system of forces

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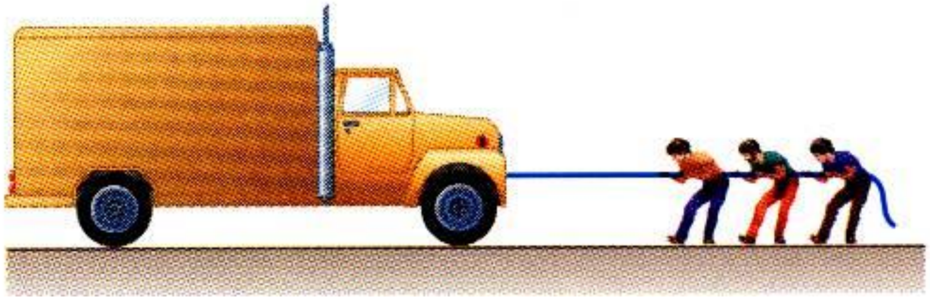
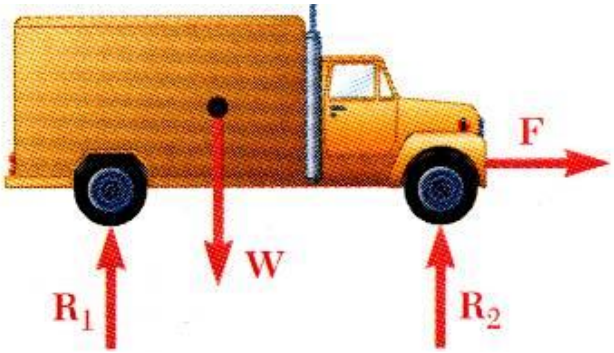
Courtesy: Vector Mechanics for Engineers, Beer and Johnston

External and Internal Forces

- Forces acting on rigid bodies are divided into two groups:
  - External forces
  - Internal forces



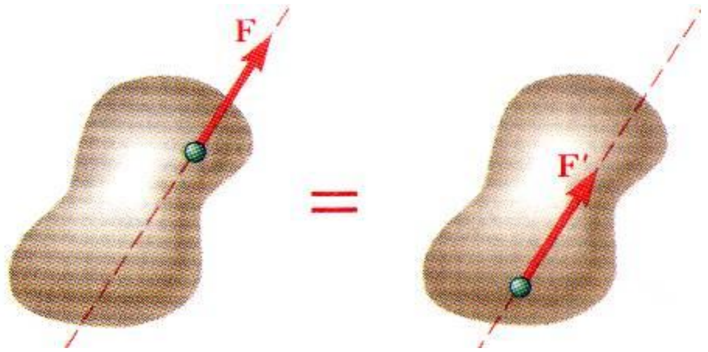
- External forces are shown in a free-body diagram.



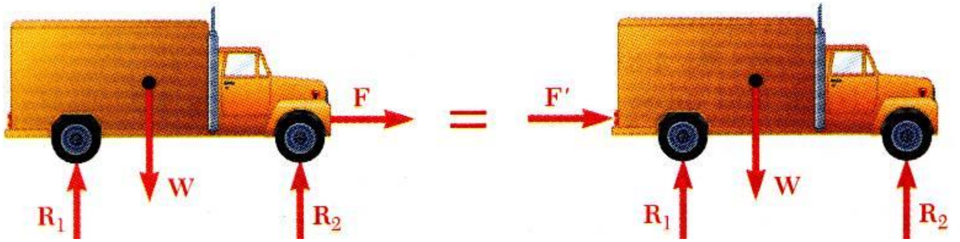
- If unopposed, each external force can impart a motion of translation or rotation, or both.

Principle of Transmissibility: Equivalent Forces

- *Principle of Transmissibility* - Conditions of equilibrium or motion are not affected by *transmitting* a force along its line of action.
  NOTE: **F** and **F'** are equivalent forces.



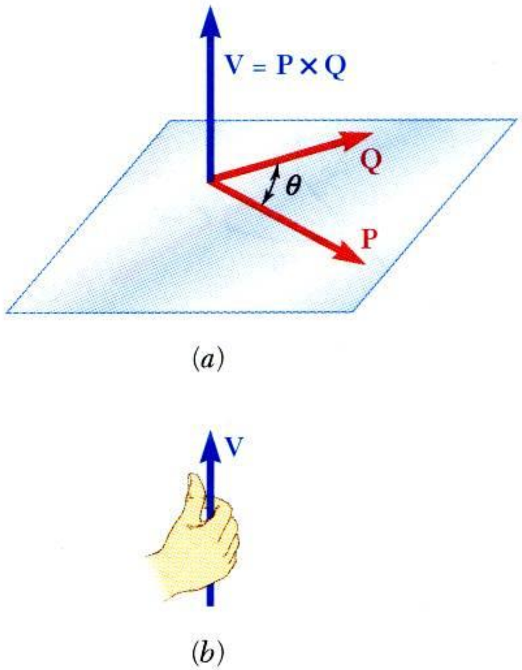
- Moving the point of application of the force **F** to the rear bumper does not affect the motion or the other forces acting on the truck.



Vector Product of Two Vectors

- Concept of the moment of a force about a point is more easily understood through applications of the *vector product* or *cross product*.

- Vector product of two vectors $\boldsymbol{P}$ and $\boldsymbol{Q}$ is defined as the vector $\boldsymbol{V}$ which satisfies the following conditions:
  1. Line of action of $\boldsymbol{V}$ is perpendicular to plane containing $\boldsymbol{P}$ and $\boldsymbol{Q}$.
  2. Magnitude of $\boldsymbol{V}$ is $V = PQ\sin\theta$
  3. Direction of $\boldsymbol{V}$ is obtained from the right-hand rule.

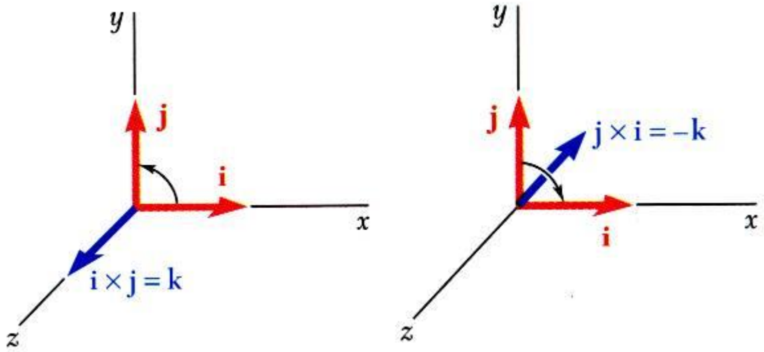


- Vector products:
  - are not commutative, $\boldsymbol{Q}\times\boldsymbol{P} = -(\boldsymbol{P}\times\boldsymbol{Q})$
  - are distributive, $\boldsymbol{P}\times(\boldsymbol{Q}_1+\boldsymbol{Q}_2) = \boldsymbol{P}\times\boldsymbol{Q}_1 + \boldsymbol{P}\times\boldsymbol{Q}_2$
  - are not associative, $(\boldsymbol{P}\times\boldsymbol{Q})\times\boldsymbol{S} \neq \boldsymbol{P}\times(\boldsymbol{Q}\times\boldsymbol{S})$

Vector Products: Rectangular Components

- Vector products of Cartesian unit vectors,

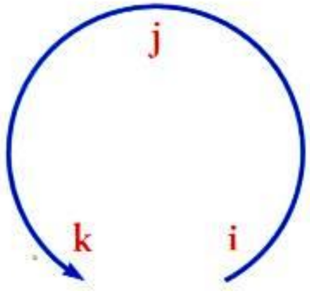
$$\vec{i}\times\vec{i}=0 \qquad \vec{j}\times\vec{i}=-\vec{k} \qquad \vec{k}\times\vec{i}=\vec{j}$$
$$\vec{i}\times\vec{j}=\vec{k} \qquad \vec{j}\times\vec{j}=0 \qquad \vec{k}\times\vec{j}=-\vec{i}$$
$$\vec{i}\times\vec{k}=-\vec{j} \qquad \vec{j}\times\vec{k}=\vec{i} \qquad \vec{k}\times\vec{k}=0$$



- Vector products in terms of rectangular coordinates

$$\vec{V}=\left(P_x\vec{i}+P_y\vec{j}+P_z\vec{k}\right)\times\left(Q_x\vec{i}+Q_y\vec{j}+Q_z\vec{k}\right)$$
$$=\left(P_yQ_z-P_zQ_y\right)\vec{i}+\left(P_zQ_x-P_xQ_z\right)\vec{j}$$
$$+\left(P_xQ_y-P_yQ_x\right)\vec{k}$$

$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ P_x & P_y & P_z\\ Q_x & Q_y & Q_z\end{vmatrix}$$



Moment of a Force About a Point

- A force vector is defined by its magnitude and direction. Its effect on the rigid body also depends on it point of application.

- The *moment* of $\boldsymbol{F}$ about O is defined as

$$\boldsymbol{M_O} = \boldsymbol{r} \times \boldsymbol{F}$$

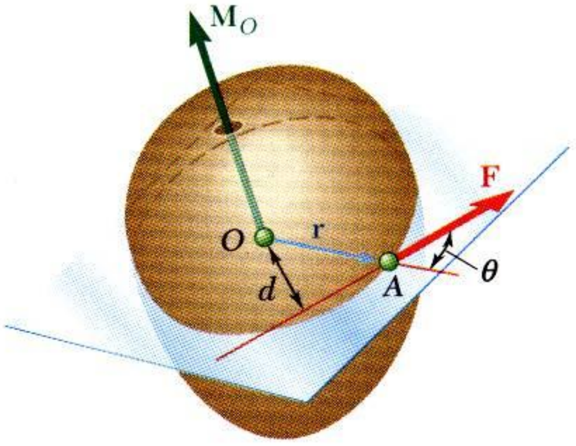
- The moment vector $\boldsymbol{M_O}$ is perpendicular to the plane containing O and the force $\boldsymbol{F}$.

- Magnitude of $\boldsymbol{M_O}$ measures the tendency of the force to cause rotation of the body about an axis along $\boldsymbol{M_O}$.

$$M_O = rF \sin\theta = Fd$$

  The sense of the moment may be determined by the right-hand rule.

- Any force $\boldsymbol{F'}$ that has the same magnitude and direction as $\boldsymbol{F}$, is *equivalent* if it also has the same line of action and therefore, produces the same moment.



(a)

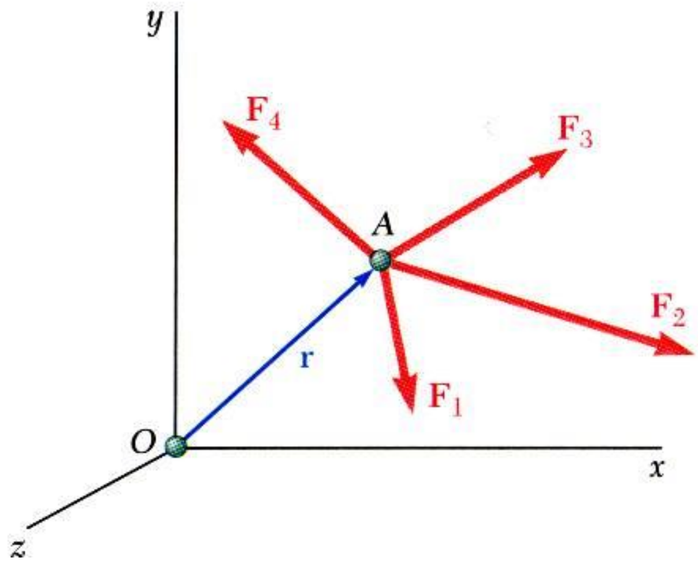
(b)

Varignon's Theorem

- The moment about a give point *O* of the resultant of several concurrent forces is equal to the sum of the moments of the various moments about the same point *O*.

$$\vec{r} \times \left(\vec{F}_1 + \vec{F}_2 + \cdots\right) = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2 + \cdots$$

- Varigon's Theorem makes it possible to replace the direct determination of the moment of a force ***F*** by the moments of two or more component forces of ***F***.

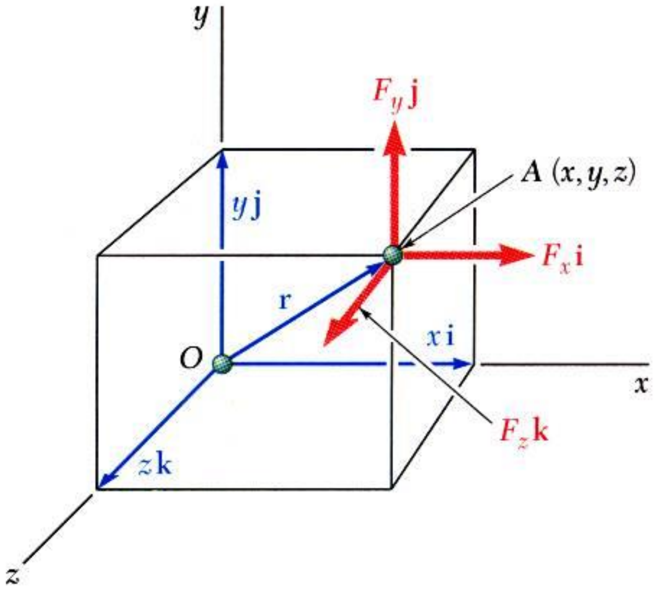


Rectangular Components of the Moment of a Force

The moment of ***F*** about *O*,

$$\vec{M}_O = \vec{r} \times \vec{F}, \qquad \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$



$$\vec{M}_O = M_x\vec{i} + M_y\vec{j} + M_z\vec{k}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(yF_z - zF_y\right)\vec{i} + \left(zF_x - xF_z\right)\vec{j} + \left(xF_y - yF_x\right)\vec{k}$$

Rectangular Components of the Moment of a Force

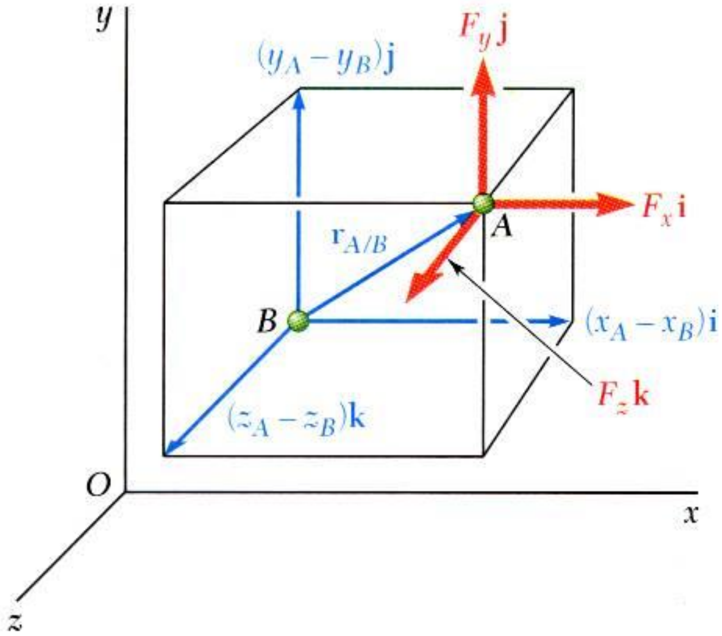
The moment of ***F*** about *B*,

$$\vec{M}_B = \vec{r}_{A/B} \times \vec{F}$$

$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

$$= (x_A - x_B)\vec{i} + (y_A - y_B)\vec{j} + (z_A - z_B)\vec{k}$$

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$



$$\vec{M}_B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ (x_A - x_B) & (y_A - y_B) & (z_A - z_B) \\ F_x & F_y & F_z \end{vmatrix}$$

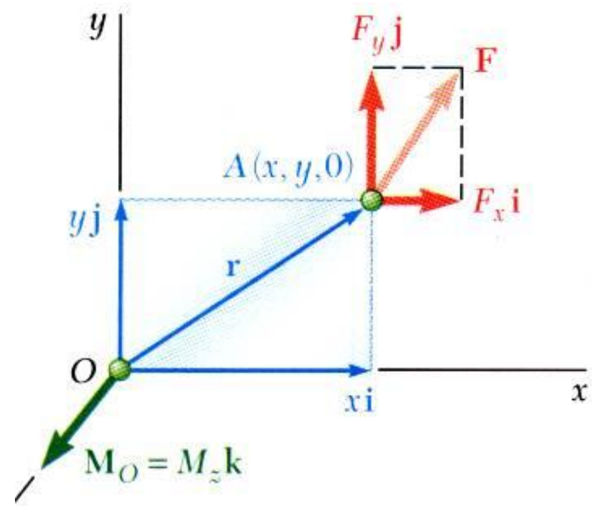
Rectangular Components of the Moment of a Force

For two-dimensional structures,

$$\vec{M}_O = \left(xF_y - yF_z\right)\vec{k}$$

$$M_O = M_Z$$

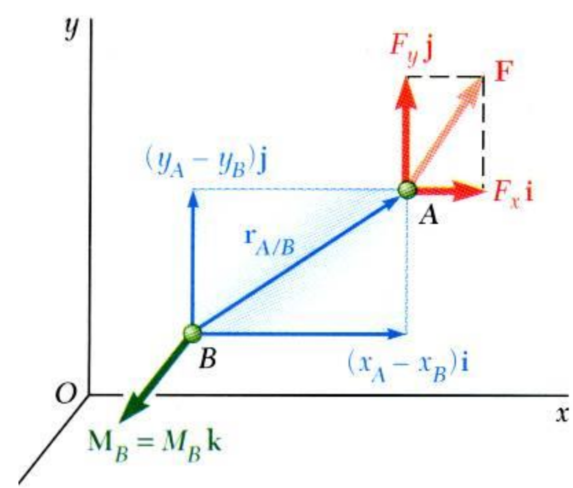
$$= xF_y - yF_z$$



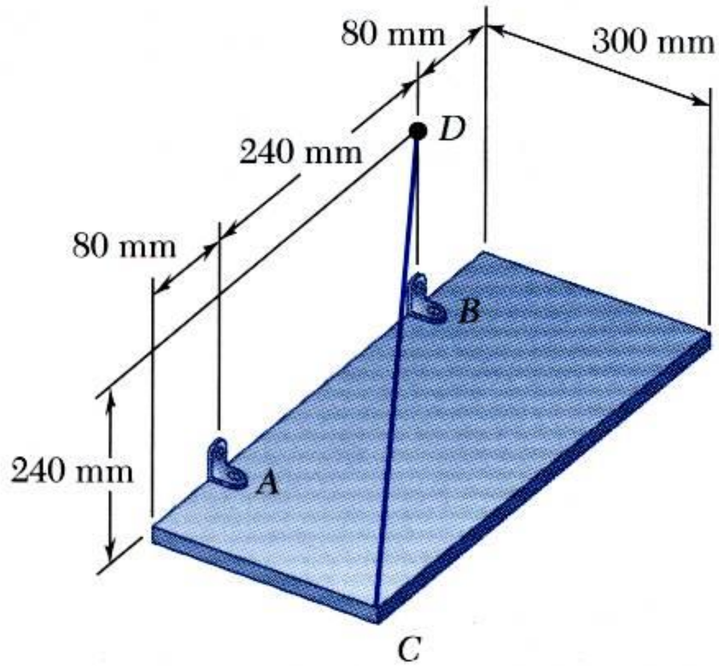
$$\vec{M}_O = \left[(x_A - x_B)F_y - (y_A - y_B)F_z\right]\vec{k}$$

$$M_O = M_Z$$

$$= (x_A - x_B)F_y - (y_A - y_B)F_z$$



Problem 3.4



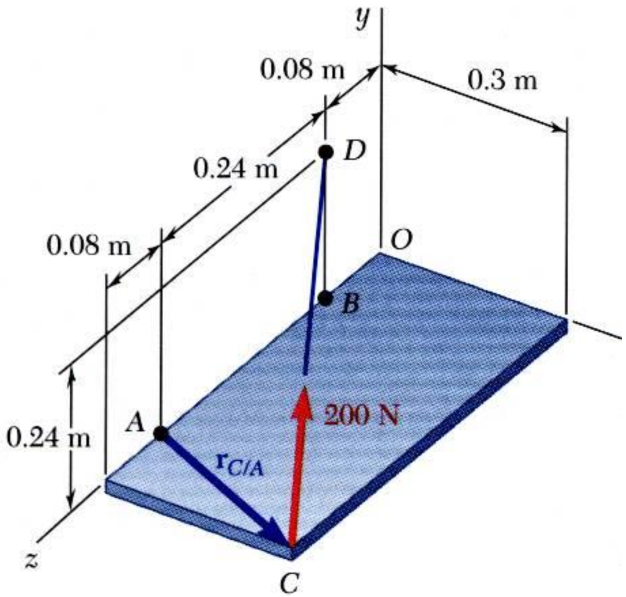
The rectangular plate is supported by the brackets at *A* and *B* and by a wire *CD*. Knowing that the tension in the wire is 200 N, determine the moment about *A* of the force exerted by the wire at *C*.

SOLUTION:

The moment M_A of the force F exerted by the wire is obtained by evaluating the vector product,

$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

Problem 3.4



SOLUTION:

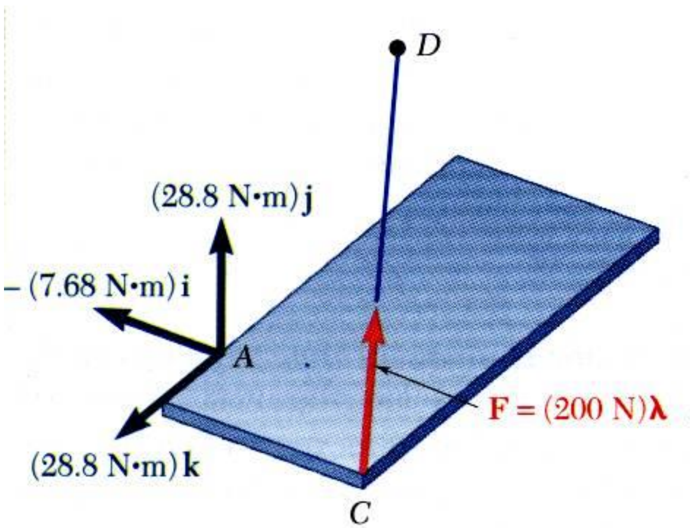
$$\vec{M}_A = \vec{r}_{C/A} \times \vec{F}$$

$$\vec{r}_{C/A} = \vec{r}_C - \vec{r}_A = (0.3\,\text{m})\vec{i} + (0.08\,\text{m})\vec{k}$$

$$\vec{F} = F\vec{\lambda} = (200\text{ N})\frac{\vec{r}_{D/C}}{r_{D/C}}$$

$$= (200\text{ N})\frac{-(0.3\,\text{m})\vec{i} + (0.24\,\text{m})\vec{j} - (0.32\,\text{m})\vec{k}}{0.5\,\text{m}}$$

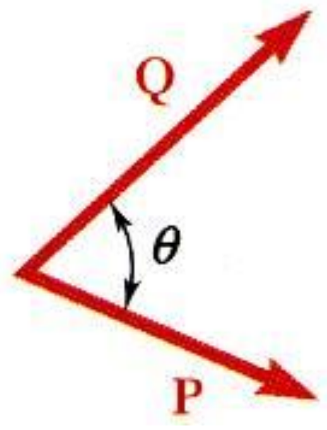
$$= -(120\text{ N})\vec{i} + (96\text{ N})\vec{j} - (128\text{ N})\vec{k}$$



$$\vec{M}_A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0.3 & 0 & 0.08 \\ -120 & 96 & -128 \end{vmatrix}$$

$$\boxed{\vec{M}_A = -(7.68\text{ N}\cdot\text{m})\vec{i} + (28.8\text{ N}\cdot\text{m})\vec{j} + (28.8\text{ N}\cdot\text{m})\vec{k}}$$

Scalar Product of Two Vectors



- The *scalar product* or *dot product* between two vectors ***P*** and ***Q*** is defined as

  $\vec{P} \bullet \vec{Q} = PQ\cos\theta \quad (\text{scalar result})$

- Scalar products:
  - are commutative, $\vec{P} \bullet \vec{Q} = \vec{Q} \bullet \vec{P}$
  - are distributive, $\vec{P} \bullet \left(\vec{Q}_1 + \vec{Q}_2\right) = \vec{P} \bullet \vec{Q}_1 + \vec{P} \bullet \vec{Q}_2$
  - are not associative, $\left(\vec{P} \bullet \vec{Q}\right) \bullet \vec{S} = \text{undefined}$

- Scalar products with Cartesian unit components,

$$\vec{P} \bullet \vec{Q} = \left(P_x\vec{i} + P_y\vec{j} + P_z\vec{k}\right) \bullet \left(Q_x\vec{i} + Q_y\vec{j} + Q_z\vec{k}\right)$$

$$\vec{i} \bullet \vec{i} = 1 \quad \vec{j} \bullet \vec{j} = 1 \quad \vec{k} \bullet \vec{k} = 1 \quad \vec{i} \bullet \vec{j} = 0 \quad \vec{j} \bullet \vec{k} = 0 \quad \vec{k} \bullet \vec{i} = 0$$

$$\vec{P} \bullet \vec{Q} = P_xQ_x + P_yQ_y + P_zQ_z$$

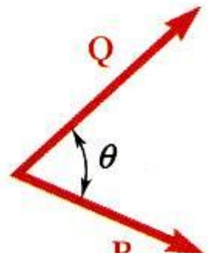
$$\vec{P} \bullet \vec{P} = P_x^2 + P_y^2 + P_z^2 = P^2$$

Scalar Product of Two Vectors: Applications

- Angle between two vectors:

$$\vec{P} \bullet \vec{Q} = PQ\cos\theta = P_x Q_x + P_y Q_y + P_z Q_z$$

$$\cos\theta = \frac{P_x Q_x + P_y Q_y + P_z Q_z}{PQ}$$

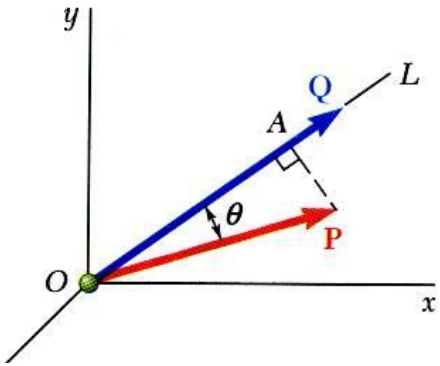


- Projection of a vector on a given axis:

$$P_{OL} = P\cos\theta = \text{projection of } P \text{ along } OL$$

$$\vec{P} \bullet \vec{Q} = PQ\cos\theta$$

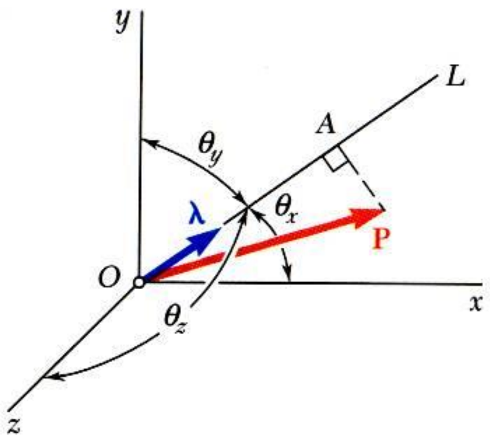
$$\frac{\vec{P} \bullet \vec{Q}}{Q} = P\cos\theta = P_{OL}$$



- For an axis defined by a unit vector:

$$P_{OL} = \vec{P} \bullet \vec{\lambda}$$

$$= P_x \cos\theta_x + P_y \cos\theta_y + P_z \cos\theta_z$$



Moment of a Force About a Given Axis

- Moment $\boldsymbol{M_O}$ of a force $\boldsymbol{F}$ applied at the point $\boldsymbol{A}$ about a point $\boldsymbol{O}$,

$$\vec{M}_O = \vec{r} \times \vec{F}$$

- Scalar moment M_{OL} about an axis $\boldsymbol{OL}$ is the projection of the moment vector $\boldsymbol{M_O}$ onto the axis,

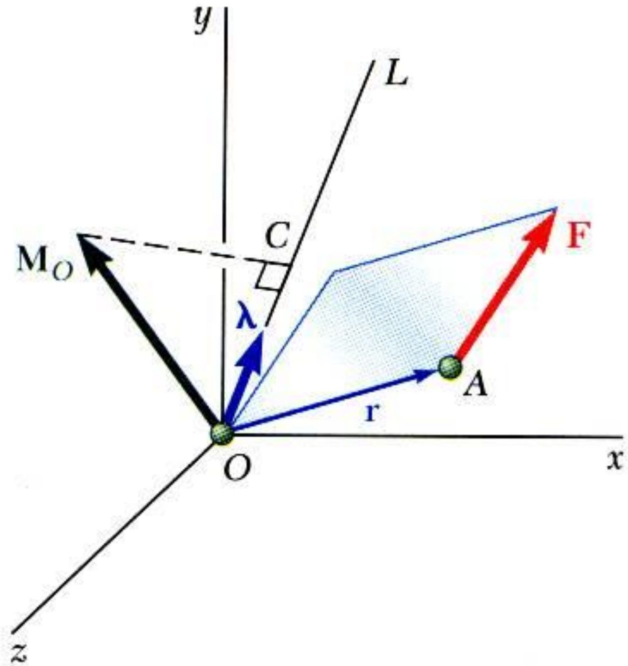
$$M_{OL} = \vec{\lambda} \bullet \vec{M}_O = \vec{\lambda} \bullet \left(\vec{r} \times \vec{F}\right)$$

- Moments of $\boldsymbol{F}$ about the coordinate axes,

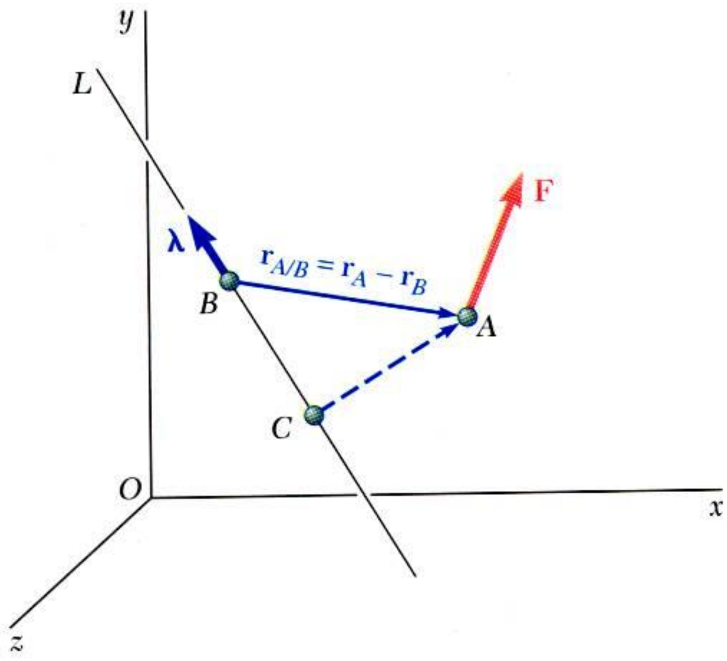
$$M_x = yF_z - zF_y$$

$$M_y = zF_x - xF_z$$

$$M_z = xF_y - yF_x$$



Moment of a Force About a Given Axis



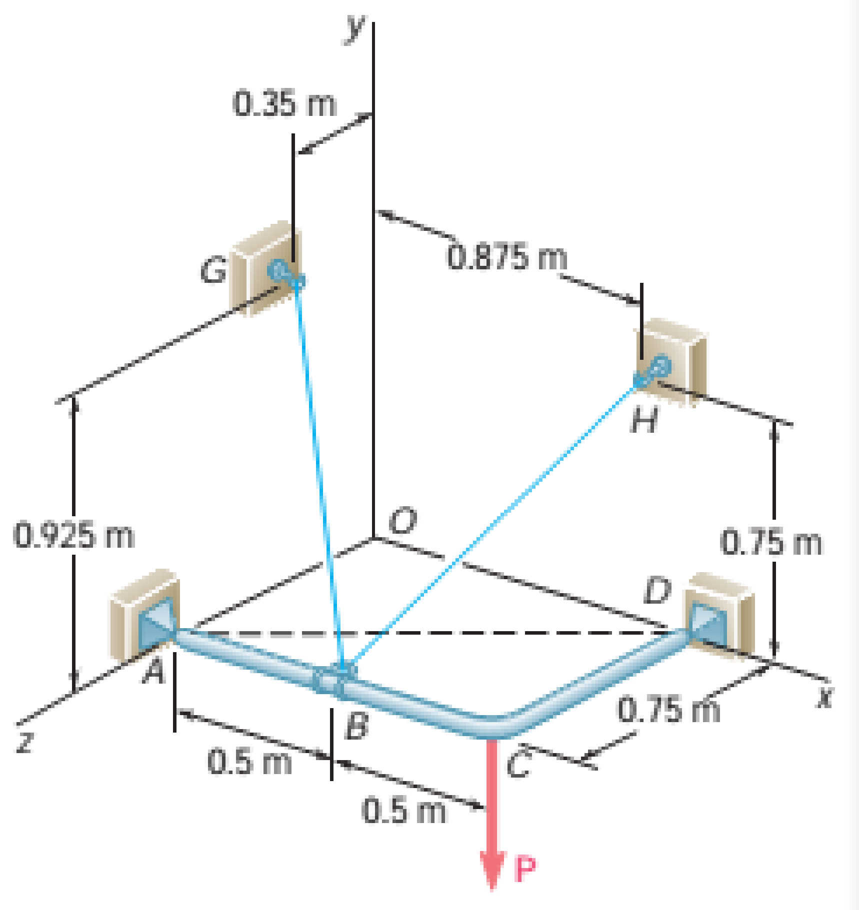
- Moment of a force about an arbitrary axis,

$$M_{BL} = \vec{\lambda} \bullet \vec{M}_B$$
$$= \vec{\lambda} \bullet \left(\vec{r}_{A/B} \times \vec{F} \right)$$
$$\vec{r}_{A/B} = \vec{r}_A - \vec{r}_B$$

- The result is independent of the point *B* along the given axis.

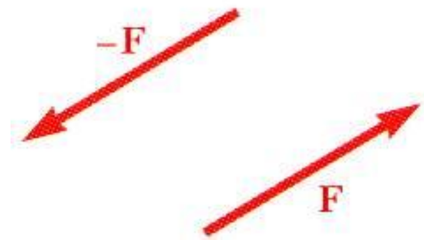
Problem# 3.59

The frame ACD is hinged at A and D and is supported by a cable that passes through a ring at B and is attached to hooks at G and H. Knowing that the tension in the cable is 450 N, determine the moment about the diagonal AD of the force exerted on the frame by portion BH of the cable.



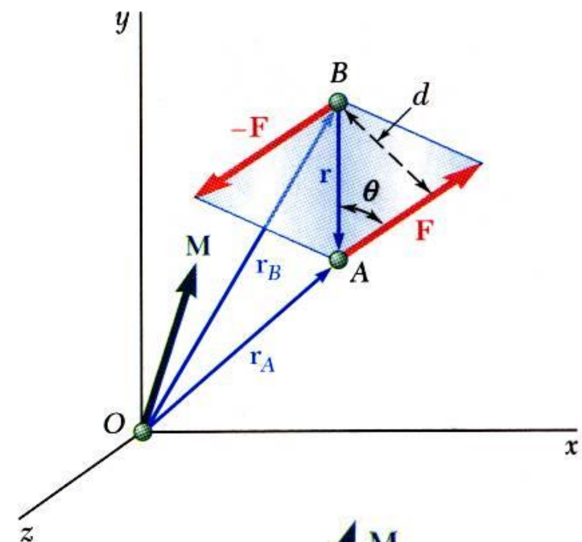
Moment of a Couple

- Two forces ***F*** and ***-F*** having the same magnitude, parallel lines of action, and opposite sense are said to form a *couple*.

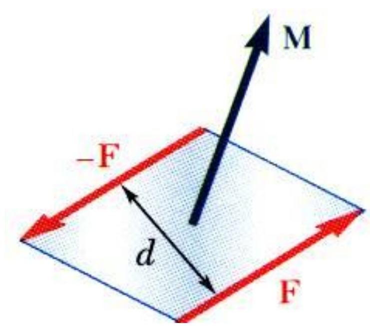


- Moment of the couple,

$$\vec{M} = \vec{r}_A \times \vec{F} + \vec{r}_B \times \left(-\vec{F}\right)$$
$$= \left(\vec{r}_A - \vec{r}_B\right) \times \vec{F}$$
$$= \vec{r} \times \vec{F}$$
$$M = rF\sin\theta = Fd$$



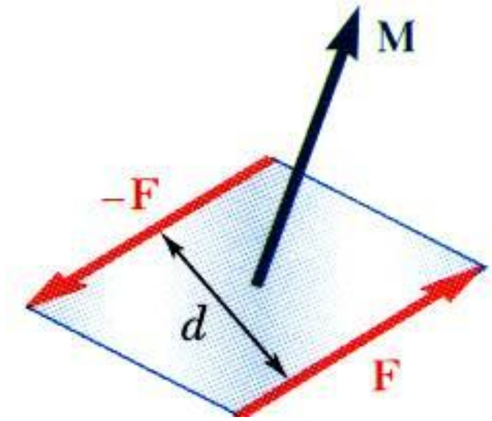
- The moment vector of the couple is independent of the choice of the origin of the coordinate axes, i.e., it is a *free vector* that can be applied at any point with the same effect.

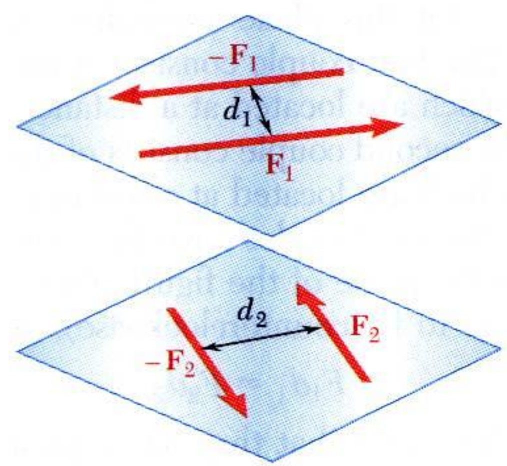


Moment of a Couple

Two couples will have equal moments if

- $F_1 d_1 = F_2 d_2$
- the two couples lie in parallel planes, and
- the two couples have the same sense or the tendency to cause rotation in the same direction.



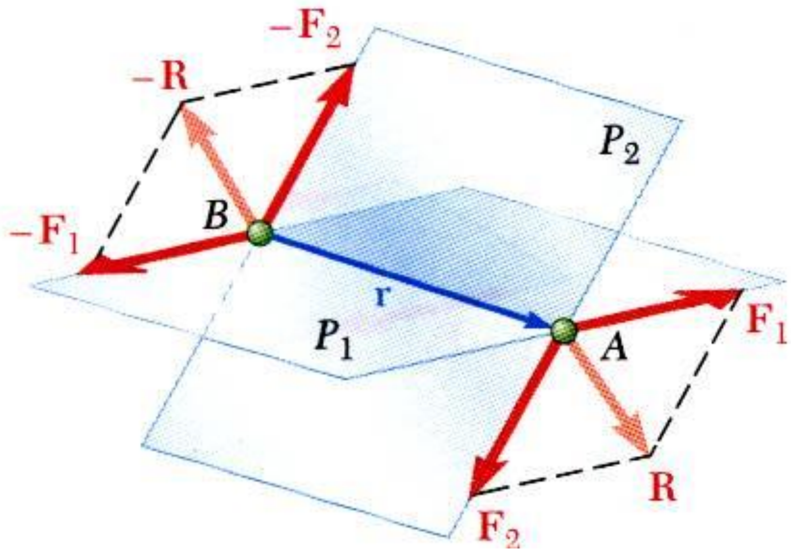


Addition of Couples

- Consider two intersecting planes P_1 and P_2 with each containing a couple

  $\vec{M}_1 = \vec{r} \times \vec{F}_1$ in plane P_1

  $\vec{M}_2 = \vec{r} \times \vec{F}_2$ in plane P_2

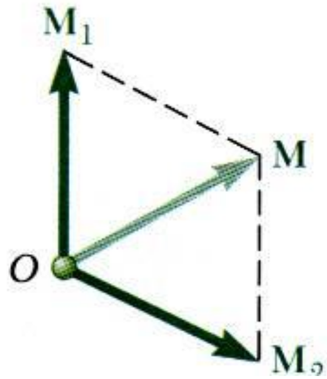


- Resultants of the vectors also form a couple

  $$\vec{M} = \vec{r} \times \vec{R} = \vec{r} \times \left(\vec{F}_1 + \vec{F}_2\right)$$

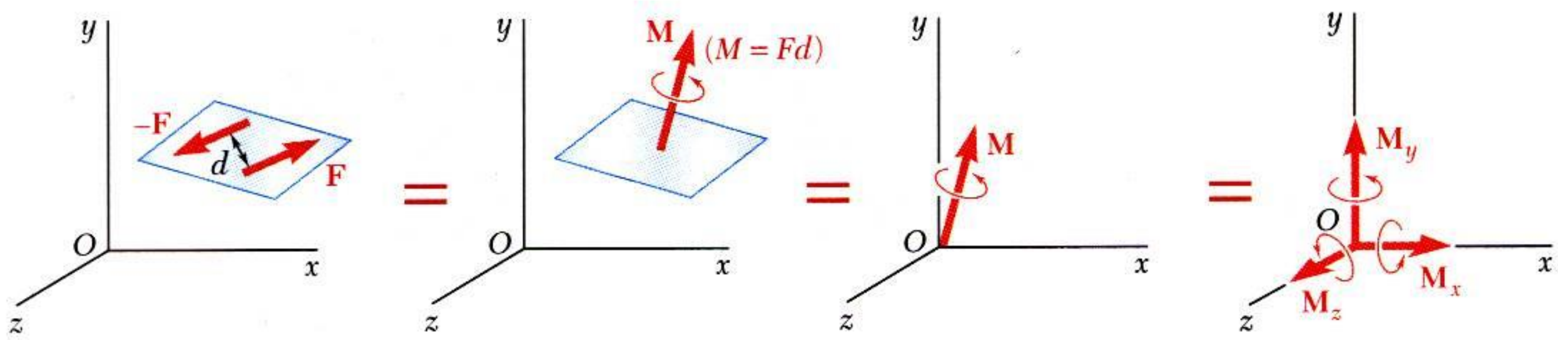
- By Varigon's theorem

  $$\vec{M} = \vec{r} \times \vec{F}_1 + \vec{r} \times \vec{F}_2$$
  $$= \vec{M}_1 + \vec{M}_2$$



- Sum of two couples is also a couple that is equal to the vector sum of the two couples

Couples Can Be Represented by Vectors



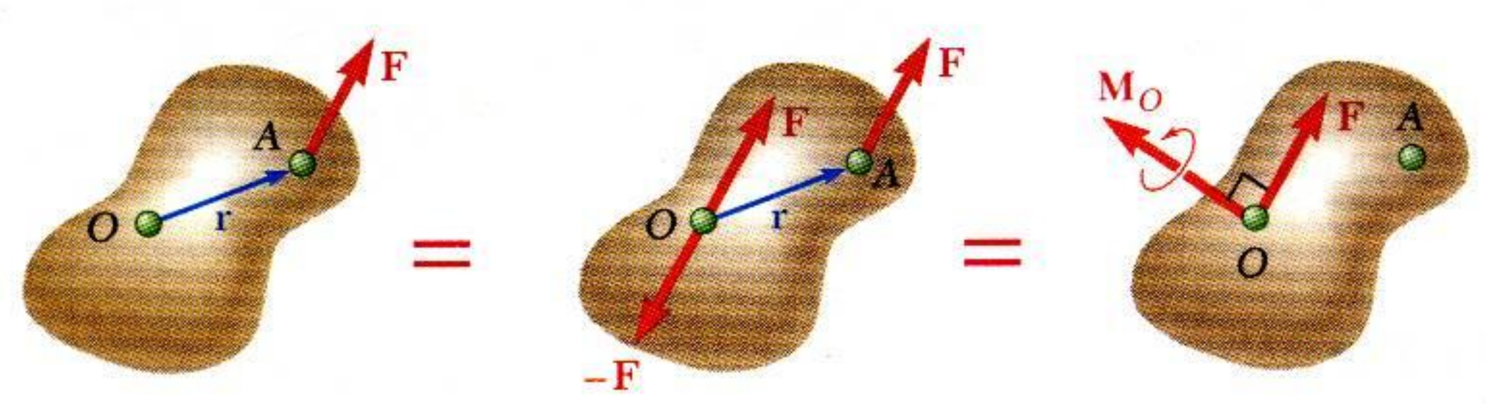
- A couple can be represented by a vector with magnitude and direction equal to the moment of the couple.

- *Couple vectors* obey the law of addition of vectors.

- Couple vectors are free vectors, i.e., the point of application is not significant.

- Couple vectors may be resolved into component vectors.

Resolution of a Force Into a Force and a Couple



- Force vector ***F*** can not be simply moved to *O* without modifying its action on the body.
- Attaching equal and opposite force vectors at *O* produces no net effect on the body.
- The three forces may be replaced by an equivalent force vector and couple vector, i.e, a *force-couple system*.

Prob#3.87

Three control rods attached to a lever ABC exert on it the forces shown.

(a) Replace the three forces with an equivalent force-couple system at B.

(b) Determine the single force that is equivalent to the force-couple system obtained in part (a), and specify its point of application on the lever.

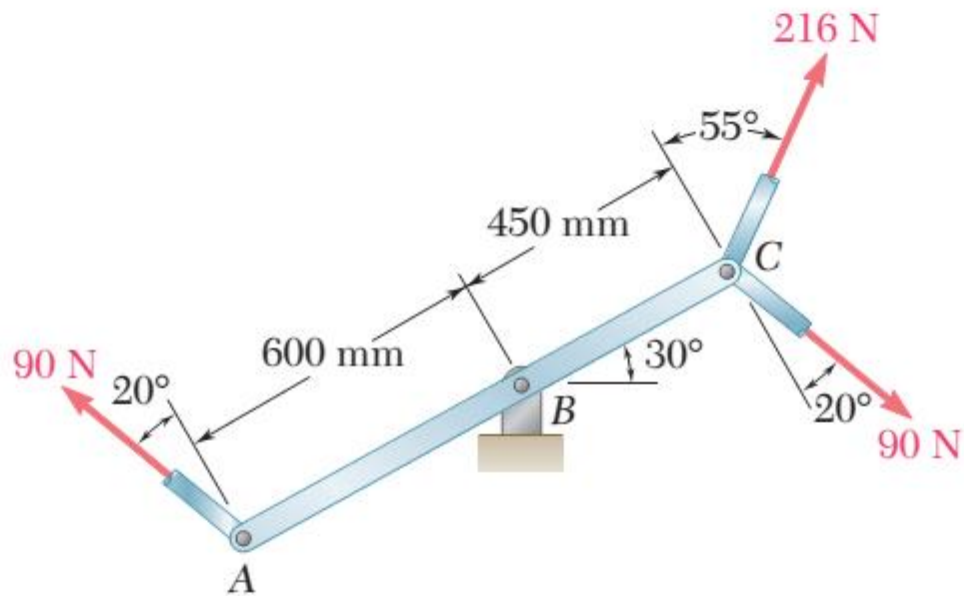
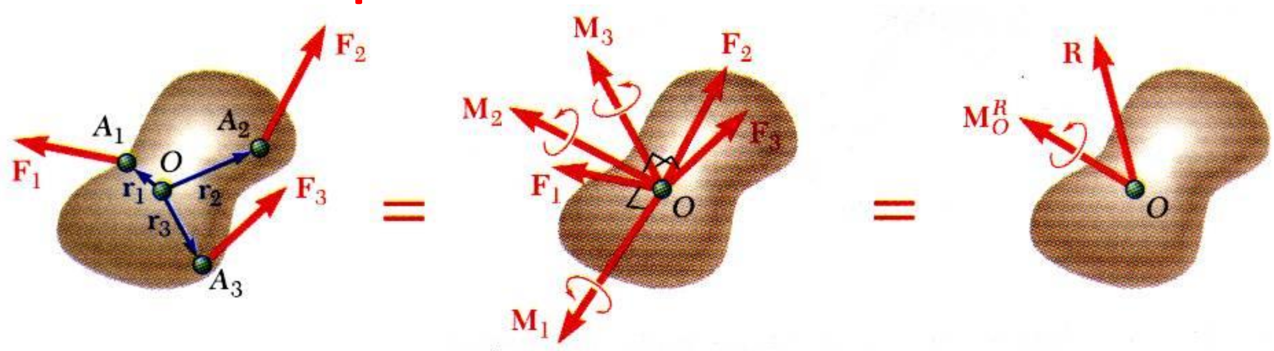
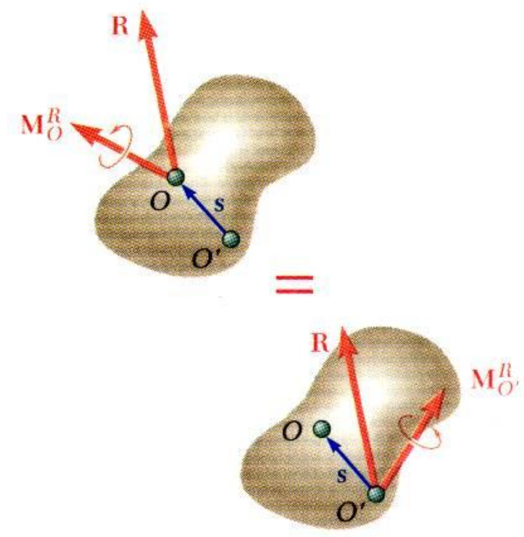


Fig. P3.87

System of Forces: Reduction to a Force and Couple





- A system of forces may be replaced by a collection of force-couple systems acting a given point *O*

- The force and couple vectors may be combined into a resultant force vector and a resultant couple vector,

$$\vec{R} = \sum \vec{F} \qquad \vec{M}_O^R = \sum \left(\vec{r} \times \vec{F} \right)$$

- The force-couple system at *O* may be moved to *O'* with the addition of the moment of ***R*** about *O'* ,

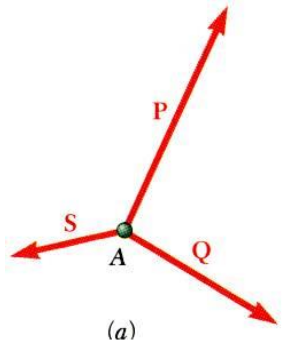
$$\vec{M}_{O'}^R = \vec{M}_O^R + \vec{s} \times \vec{R}$$

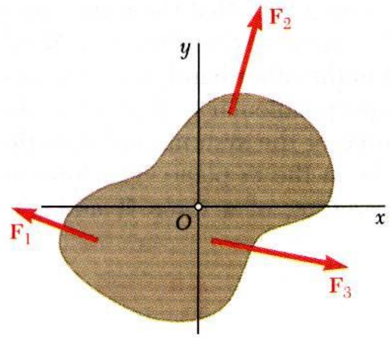
- Two systems of forces are equivalent if they can be reduced to the same force-couple system.

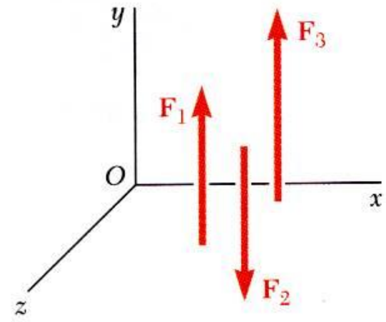
Further Reduction of a System of Forces

- If the resultant force and couple at *O* are mutually perpendicular, they can be replaced by a single force acting along a new line of action.

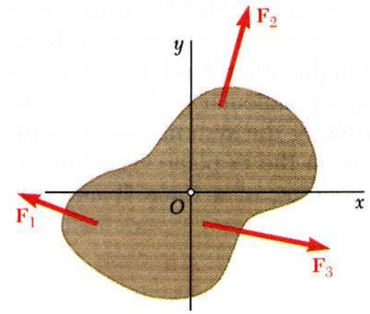
- The resultant force-couple system for a system of forces will be mutually perpendicular if:
  1) the forces are concurrent,
  2) the forces are coplanar, or
  3) the forces are parallel.

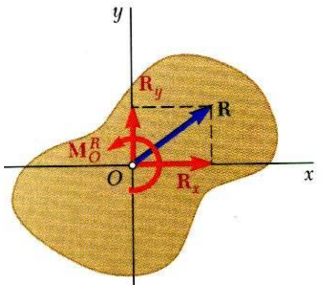




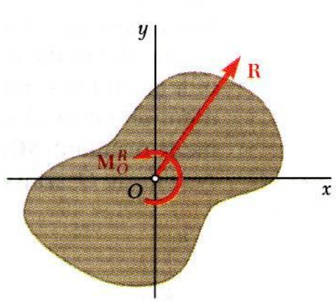


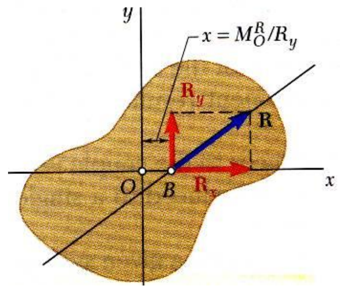
Further Reduction of a System of Forces





- System of coplanar forces is reduced to a force-couple system $\vec{R}$ and $\vec{M}_O^R$ that is mutually perpendicular.

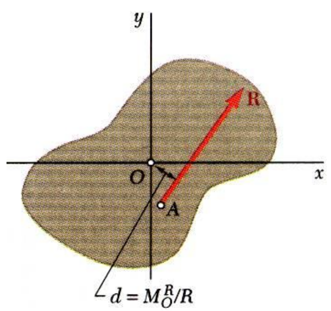


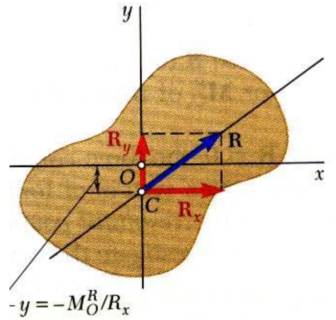


- System can be reduced to a single force by moving the line of action of $\vec{R}$ until its moment about O becomes $\vec{M}_O^R$

- In terms of rectangular coordinates,

$$xR_y - yR_x = M_O^R$$





Prob# 3.111

A machine component is subjected to the forces and couples shown. The component is to be held in place by a single rivet that can resist a force but not a couple. For P = 0, determine the location of the rivet hole if it is to be located on line FG,

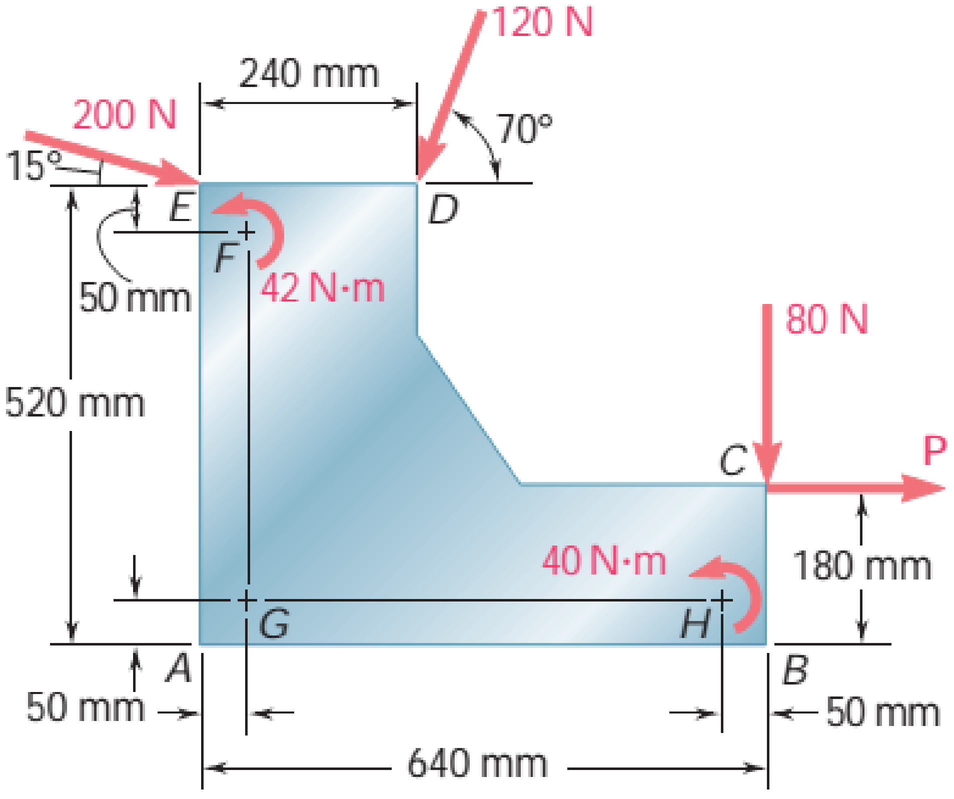


Fig. P3.111